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Success and Limits of Mathematization

The two conspicuous traits of mathematics are, first, precision, and, second, the availability of algorithms and rigorous proofs. We regiment a technical language with a view to achieving the most efficient formulation we can of the regularities that hold good of the subject matter; and in some cases this effort produces an algorithm, rendering the laws recognizable by computation. In other cases one settles for a proof procedure, consisting perhaps of a compact codification of so-called axioms and some rules for generating further laws from them.

Mathematical language is the far extreme of this sort of progress. Mathematization is what this progress may be called, if only in its farther reaches.

There has been a perverse tendency to think of mathematics primarily as abstract or uninterpreted and only secondarily as interpreted or applied, and then to philosophize about application. This was the attitude of Russell at the turn of the century, when he wrote that in pure mathematics "we never know what we are talking about, nor whether what we are saying is true."[1] He expressed the same attitude less

This piece, plus two initial pages here omitted, was my contribution to a symposium under this title at the sixteenth International Congress of Philosophy, Düsseldorf, 1978.

1. *Mysticism and Logic*, p. 75. The passage dates from 1901.

wittily thus: "Pure mathematics is the class of all propositions of the form '*p* implies *q*', where *p* and *q* are propositions containing one or more variables, the same in the two propositions, and neither '*p*' nor '*q*' contains any constants except logical constants."[2] On this view all that is left to the mathematician, for him to be right or wrong about, is whether various of his uninterpreted sentence schemata follow logically from his uninterpreted axiom schemata. All that is left to him is elementary logic, the first-order predicate calculus.

This disinterpretation of mathematics was a response to non-Euclidean geometry. Geometries came to be seen as a family of uninterpreted systems. The first geometry to be studied was indeed abstracted from the technology of architecture and surveying in ancient Egypt, but it is to be reckoned as pure mathematics only after disinterpretation; such was the new view. From geometry the view spread to mathematics generally.

What then of elementary arithmetic? Pure number, pure addition, and the rest would be viewed as uninterpreted; and their application, then, say to apples, would consist perhaps in interpreting the numbers five and twelve as piles of apples, and addition as piling them together.

I find this attitude perverse. The words 'five' and 'twelve' are at no point uninterpreted; they are as integral to our interpreted language as the word 'apple' itself. They name two intangible objects, numbers, which are *sizes of* sets of apples and the like. The 'plus' of addition is likewise interpreted from start to finish, but it has nothing to do with piling things together. Five plus twelve is how many apples there are in two separate piles of five and twelve, without their being piled together.

The expressions 'five', 'twelve', and 'five plus twelve' differ from 'apple' in not denoting bodies, but this is no cause for disinterpretation; the same can be said of such unmathematical terms as 'nation' or 'species'. Ordinary interpreted scientific discourse is as irredeemably committed to abstract objects—to nations, species, numbers, functions, sets—as it

2. *Principles of Mathematics*, p. 3.

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is to apples and other bodies. All these things figure as values of the variables in our overall system of the world. The numbers and functions contribute just as genuinely to physical theory as do hypothetical particles.

Arithmetic is a paragon, certainly, of the mathematical virtues. Its terms are precise and they lend themselves to admirable algorithms. But these virtues were achieved through the progressive sharpening and regimenting of terms and idioms while they remained embedded in the regular interpreted language. Arithmetic is related to unregimented language in the same way as is the logic of truth functions; there is no call for disinterpretation followed by application. The case of set theory, again, is similar; it comes of a sharpening and regimenting of ordinary talk of properties or classes. Arithmetic, logic, and set theory are purely mathematical, but their purity has nothing to do with disinterpretation; all it means is that the arithmetical, logical, and set-theoretic techniques are formulated without recourse to locutions from outside the arithmetical or logical or set-theoretic part of our general vocabulary. Purity is not uninterpretedness.

A progressive sharpening and regimenting of ordinary idioms: this is what led to arithmetic, symbolic logic, and set theory, and this is mathematization. Once it has been achieved by arduous evolution in one domain, it may sometimes be achieved swiftly in another domain by analogy; for the mathematical notation that was developed in one domain may, by *re*interpretation, be put to use in another. A simple example is the reinterpretation of truth functions as electric circuits. An even simpler example is the use of graphs in economics and elsewhere. Geometry, to begin with, is a sharpening and regimenting of existing idioms regarding physical space, the space of taut strings and light rays and trajectories; by reinterpretation, afterward, what had originally designated a curve in physical space might be reinterpreted as expressing a relation between supply and demand, or between employment and national product, or between the sine of an angle and the size of the angle. These analogical reinterpretations have fostered the unfortunate conception of mathematics as basically uninterpreted.

Analogy also takes another line. After some subject matter has been well mathematized and has come to enjoy a smooth algorithm, the mathematician may construct another and this time genuinely uninterpreted system in *partial* analogy. He may do so by denying one of the component laws, or by generalizing on some special feature. Such was the origin of the non-Euclidean geometries and *n*-dimensional geometry. Systematic variation of this sort, on a wholesale basis, is the business of abstract algebra. Some of the systems thus produced find useful interpretations afterward, but the driving force is not that; it is intellectual curiosity regarding the structures themselves. There is thus no denying the magnitude of the role played in modern mathematics by uninterpreted systems. It is the tail that has come to wag the dog. What I was deploring, however, in deploring the all too popular view represented by the early Russell, was the failure to recognize the existence—let alone the philosophical importance—of the little old dog itself.

In a higher sense, even abstract algebra and the abstract geometrical studies may be said to be fully interpreted studies after all; they are chapters of set theory. A group, for instance, is simply a function of a certain sort. It is any associative two-place function having a unique identity element and for each element an inverse. But a two-place function is a set of triples, and thus group theory is the part of set theory that explores the properties common to functions that meet these conditions. Other abstract algebras can be identified with other set-theoretic structures in a similar spirit.

Mathematics can stand aloof from application to natural science also without being uninterpreted. Higher set theory is a striking case of this. I already urged that set theory, arithmetic, and symbolic logic are all of them products of the straightforward mathematization of ordinary interpreted discourse—mathematization *in situ*. Set-theoretic laws come of regimenting the ways of reasoning about classes or properties, ways of reasoning that already prevailed more or less tacitly in natural science and ordinary discourse. More particularly, as it happens, this regimentation has been a matter of clearing away implicit contradictions. Once the laws are formulated, however, along as simple and general lines as we can manage, we find that they are rich also in implications that outrun any past or contemplated uses, implications regarding infinite sets and transfinite numbers. Bifurcations emerge, moreover, over the axiom of choice or the continuum hypothesis or the existence of inaccessible numbers, where there is a free option between alternative principles without there being any effect on applications in natural science. Mathematicians are driven to pursue these matters by the same disinterested intellectual curiosity that impels them into abstract algebras and odd geometries; yet in this case, unlike those, there has been no departure from interpreted theory.

The branch of mathematics that is most widely and conspicuously used is elementary arithmetic. Next come the parts of mathematics that are built on arithmetic: the algebra of real and complex numbers, the theory of functions, the differential and integral calculus. The ubiquitous use of elementary arithmetic was to be expected, since all sorts of things can be counted and many of them are worth counting. After counting comes measurement. A great invention, measurement; it enables us to compare amounts of valuable stuff that does not lend itself directly to counting. It is measurement that makes for the widespread use of the quantitative branches of mathematics beyond elementary arithmetic.

But if the need to compare amounts of valuable stuff was what fostered the invention of measurement, that use of measurement has subsequently been dwarfed by other uses. Measurement is central to natural science because of the predictive power of concomitant variation. Let us therefore turn our attention briefly to prediction, and induction.

Induction, primitively, was a mere matter of expecting that events that are similar by our lights will have sequels that are similar to one another. The larger the class of mutually similar antecedent events may be, all of which have had mutually similar sequels, the stronger is the presumption of a similar sequel the next time around. But the presumption is increased overwhelmingly if variations among the antecedent events can be correlated with variations in the sequels. For this purpose measurement is brought to

bear. Measurement is devised for some varying feature of the otherwise similar antecedent events, and also for some varying feature of the otherwise similar sequels, and a constant ratio or some other simple correlation is established between the two variations. Once this is achieved, a causal connection can no longer be doubted.

Hence the advantage, for science, of quantitative terms; and they are eagerly sought for the various branches of science. These terms and the methods of measuring will differ from branch to branch, but the purely numerical part of the apparatus will be the same for all. Hence the very general scientific utility of analysis, or quantitative mathematics.

Because of the power of these methods, and ultimately the predictive power of concomitant variation, sciences clamor to be quantitative; they clamor for something to measure. This is both good and bad. It is very good indeed if the measurable quantity can be found to play a significant role in the subject matter of the science in question. It is bad if in the quest for something to measure the scientist turns his back on the original concerns of his science and is borne away, however smoothly, on a tangent of trivialities. Ills of mathematization, as well as successes, can be laid to the quest for quantitativity.

It is in the quantitative that mathematization exerts its most overwhelming attraction. More exotic branches of mathematics, however, uninterpreted to begin with, are likewise enlisted for application now and again: topology, perhaps, or Hilbert space. In such cases again there is the duality of good and evil to reckon with. A happy mathematization can work wonders, and the hope of such gains is always the ostensible motive of mathematization. But there are other contributing drives, counter-productive ones, of which the individual himself is apt to be unaware. There is methodolatry, or the love of gadgetry: the tendency to take more satisfaction in methods than in the results. Also there is the repose, the respite from hard thought and hairy decisions, that a smooth algorithm can bring. In these ways one may be lured into problems that lend themselves to favorable techniques, though they not be the problems most central to one's concerns. The rise of the computer aggravates this danger.

We can sense these tensions already in the following humdrum example, which involves no computers and no appreciable mathematics. Amid the vague and amorphous matters confronting the social anthropologist, there are the clean-cut kinship structures. They loom large in primitive societies, and the anthropologist is glad; for they submit nicely to elementary symbolic logic, and do not need even that. Now it is good that there is this firm structure to which to relate other more important but less tangible factors. I suspect nevertheless that kinship cuts a disproportionate figure in anthropology just because of the methodological solace that it brings.

I have touched on the nature of mathematization, arguing that in its primary form it develops within a science rather than being applied from outside. It is continuous with the growth of precision, and it blossoms at last into algorithms and proof procedures. The most significant continuing force for mathematization was measurement, because of the benefits of concomitant variation. Finally I noted the danger of being seduced, by the glitter of algorithm, into mathematizing one's subject off the target. But I should say something, still, about the famous formal limits to mathematization that are intrinsic to the mathematics itself.

Building on Gödel's work, Alonzo Church and Alan Turing showed in 1936 that mathematization in the fullest sense is too much to ask even for so limited a subject as elementary logic. They proved that there can be no complete algorithm, no decision procedure, for the first-order predicate calculus. There is, of course, a complete proof procedure for that calculus. However, it follows from the Church-Turing theorem that there cannot even be a complete proof procedure for nonprovability in that calculus. From this it follows further that there cannot be a complete proof procedure for any branch of mathematics in which proof procedures can be modeled. Elementary number theory is already one such branch; hence Gödel's original incompleteness theorem.

Besides these necessary internal limitations on proof and algorithm, there is commonly also a voluntary one in the case

of a natural science. Mathematize as he will, and seek algorithms as he will, the empirical scientist is not going to aspire to an algorithm or proof procedure for the whole of his science; he would not want it if he could have it. He will want rather to keep a large class of his sentences open to the contingencies of future observation. It is only thus that his theory can claim empirical import.